6.2 Antidifferentiation by Substitution

DEFINITION Indefinite Integral

The family of all antiderivatives of a function $f(x)$ is the **indefinite integral of f with respect to x** and is denoted by $\int f(x)dx$.

If F is any function such that $F'(x) = f(x)$, then $\int f(x)dx = F(x) + C$, where C is an arbitrary constant, called the **constant of integration.**

EXAMPLE 1 Evaluating an Indefinite Integral

Evaluate $\int (x^2 - \sin x)\,dx$. $(f'(x) = x^2 - \sin x)$

$f(x) = \frac{1}{3}x^3 + \cos x + C$

Properties of Indefinite Integrals

$$\int k f(x)dx = k\int f(x)\,dx \quad \text{for any constant } k$$

$$\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

Power Formulas

$$\int u^n\,du = \frac{u^{n+1}}{n+1} + C \text{ when } n \neq -1$$

$$\int u^{-1}\,du = \int \frac{1}{u}\,du = \ln|u| + C$$
(see Example 2)

Trigonometric Formulas

$$\int \cos u\,du = \sin u + C$$

$$\int \sin u\,du = -\cos u + C$$

$$\int \sec^2 u\,du = \tan u + C$$

$$\int \csc^2 u\,du = -\cot u + C$$

$$\int \sec u \tan u\,du = \sec u + C$$

$$\int \csc u \cot u\,du = -\csc u + C$$

Exponential and Logarithmic Formulas

$$\int e^u\,du = e^u + C$$

$$\int a^u\,du = \frac{a^u}{\ln a} + C$$

$$\int \ln u\,du = u\ln u - u + C \quad \text{(See Example 2)}$$

$$\int \log_a u\,du = \int \frac{\ln u}{\ln a}\,du = \frac{u\ln u - u}{\ln a} + C$$

EXAMPLE 2 Verifying Antiderivative Formulas

Verify the antiderivative formulas:

(a) $\int u^{-1}\,du = \int \frac{1}{u}\,du = \ln|u| + C$ (b) $\int \ln u\,du = u\ln u - u + C$

when $u>0$ $\frac{d}{du} \ln u + C$

$= \frac{1}{u} + 0$

$= \frac{1}{u}$

when $u<0$ $\frac{d}{du} \ln u + C$

$= \frac{1}{u} + C$

$= \frac{1}{u}$

$\frac{d}{dx} u\ln u - u + C$

$= u \cdot \frac{1}{u} + \ln u(1) - 1 + 0$

$= 1 + \ln u - 1$

$= \ln u$

EXPLORATION 1 **Are $\int f(u)\,du$ and $\int f(u)\,dx$ the Same Thing?**

Let $u = x^2$ and let $f(u) = u^3$.

1. Find $\int f(u)\,du$ as a function of u.
2. Use your answer to question 1 to write $\int f(u)\,du$ as a function of x.
3. Show that $f(u) = x^6$ and find $\int f(u)\,dx$ as a function of x.
4. Are the answers to questions 2 and 3 the same?

1. $\int u^3\,du = \underline{\frac{1}{4}u^4 + C}$ $(\int f(u)\,du)$

2. $\frac{1}{4}(x^2)^4 + C = \frac{1}{4}x^8 + C$

3. $\int x^6\,dx$ $(\int f(u)\,dx)$ $= \frac{1}{7}x^7 + C$

EXAMPLE 3 **Paying Attention to the Differential**

Let $f(x) = x^3 + 1$ and let $u = x^2$. Find each of the following antiderivatives in terms of x:

(a) $\int f(x)\,dx$ **(b)** $\int f(u)\,du$ **(c)** $\int f(u)\,dx$

a) $\int (x^3 + 1)\,dx$

$= \frac{1}{4}x^4 + x + C$

b) $\int (u^3 + 1)\,du$

$= \frac{1}{4}u^4 + u + C$

$= \frac{1}{4}x^8 + x^2 + C$

c) $\int (u^3 + 1)\,dx$

$= \int (x^6 + 1)\,dx$

$= \frac{1}{7}x^7 + x + C$

U Substitution Activity

EXAMPLE 4 Using Substitution

Evaluate $\int \sin x \, e^{\cos x} \, dx$.

$u = \cos x$

$\frac{du}{dx} = -\sin x$

$dx = \frac{-1}{\sin x} du$

$\int \sin x \, e^{u} \cdot \frac{-1}{\sin x} du$

$= -\int e^{u} du$

$= -e^{u} + C$

$= -e^{\cos x} + C$

EXAMPLE 5 Using Substitution

Evaluate $\int x^2\sqrt{5+2x^3}\,dx$.

$$\int x^2(5+2x^3)^{1/2}\,dx$$

$u = 5+2x^3$

$\frac{du}{dx} = 6x^2$

$dx = \frac{1}{6x^2}\,du$

$$\int x^2 (u)^{1/2}\frac{1}{6x^2}\,du$$

$$= \int \frac{1}{6}u^{1/2}\,du$$

$$= \frac{1}{6}\left(\frac{2}{3}u^{3/2}\right)+C$$

$$= \frac{1}{9}(5+2x^3)^{3/2}+C$$

EXAMPLE 6 Using Substitution

Evaluate $\int \cot 7x \, dx$.

$$\int \frac{\cos 7x}{\sin 7x} \, dx$$

$u = \sin 7x$

$\frac{du}{dx} = 7\cos 7x$

$dx = \frac{1}{7\cos 7x} \, du$

$$\int \cos 7x \, \frac{1}{u} \cdot \frac{1}{7\cos 7x} du$$

$$= \int \frac{1}{7} \, \frac{1}{u} \, du$$

$$= \frac{1}{7} \ln|u| + C$$

$$= \frac{1}{7} \ln|\sin 7x| + C$$

EXAMPLE 7 Setting Up a Substitution with a Trigonometric Identity

Find the indefinite integrals. In each case you can use a trigonometric identity to set up a substitution.

(a) $\displaystyle\int \frac{dx}{\cos^2 2x}$ **(b)** $\displaystyle\int \cot^2 3x\,dx$ **(c)** $\displaystyle\int \cos^3 x\,dx$

EXAMPLE 8 **Evaluating a Definite Integral by Substitution**

Evaluate $\int_0^{\pi/3} \tan x \sec^2 x \, dx$.

EXAMPLE 9 That Absolute Value Again

Evaluate $\displaystyle\int_0^1 \frac{x}{x^2-4}\,dx.$